

Conformal Newton-Hooke algebras, Niederer's transformation and Pais-Uhlenbeck oscillator

Krzysztof Andrzejewski

Department of Computer Science,
University of Łódź,
Pomorska 149/153, 90-236 Łódź, Poland
E-mail: k-andrzejewski@uni.lodz.pl

Abstract

Dynamical systems invariant under the action of the l -conformal Newton-Hooke algebras are constructed by the method of nonlinear realizations. The relevant first order Lagrangians together with the corresponding Hamiltonians are found. The relation to the Galajinsky and Masterov [Phys. Lett. B 723 (2013) 190] approach as well as the higher derivatives formulation is discussed. The generalized Niederer's transformation are presented which relate the systems under consideration to those invariant under the action of the l -conformal Galilei algebra [Nucl. Phys. B 876 (2013) 309]. As a nice application of these results an analogue of Niederer's transformation, on the Hamiltonian level, for the Pais-Uhlenbeck oscillator is constructed.

1 Introduction

The Newton-Hooke algebra is a generalization of the Galilei one to the case of nonvanishing cosmological constant leading to the universal cosmological repulsion or attraction (see, e.g., [1, 2]). It is derived from the (anti) de Sitter algebra by the nonrelativistic contraction in a similar way as the Galilei

algebra is obtained from the Poincaré one. The main difference between Galilei and the Newton-Hooke algebra is that in the latter case the structure relations involving the generators of time and space translations yield the Galilei boosts: $[H, P_i] = \pm \frac{1}{R^2} K_i$. The positive constant R is called the characteristic time (and is related to the radius of the parent (anti) de Sitter space). The upper/lower sign above is realized in nonrelativistic spacetime with the negative/positive cosmological constant $\Lambda = \mp \frac{1}{R^2}$.

Conformal extensions of the Galilei and Newton-Hooke algebras have recently attracted considerable interest, mostly in the context of the nonrelativistic AdS/CFT correspondence. Such extensions are parameterized by a positive half-integer l [3]-[6], which justifies their name: l -conformal algebras. The dynamical realizations of the l -conformal Galilei and Newton-Hooke algebras involve, in general, higher derivatives terms (see, e.g., [7]-[14]). However, it is also possible (using the method of nonlinear realizations [15]-[17]) to construct invariant dynamics involving only second derivatives [18]-[20]. The method proposed in Refs. [18]-[20] allows for elegant and algorithmic construction of invariant dynamical equations. However, there remains an open problem if they admit Lagrangian and Hamiltonian formalism. In Ref. [21] it has been shown that this is possible for the case of the l -conformal Galilei algebra.

In the present paper, first, we apply the method developed in [21] to the case of the l -conformal Newton-Hooke algebra and we construct the invariant dynamics in terms of the first order Lagrangian and Hamiltonian formalism (Section 2 and 3). Moreover, we compare our approach with the one reported in Ref. [20] as well as with the Pais-Uhlenbeck theory (Section 4).

The second part of the paper is devoted to the problem of Niederer's-type transformations. In Section 5 we construct an analogon of the celebrated Niederer's transformation [22] for our approach, and we show that it leads to the results in [21] obtained for the l -conformal Galilei algebra. On the other hand, on the Lagrangian level, the generalization of Niederer's transformation has been also extensively studied for the Pais-Uhlenbeck system with *odd* frequencies (i.e., frequencies proportional to the consecutive odd integers); see, e.g., [23]-[25]. However, its Hamiltonian counterpart seems to be more involved due to the lack of the direct transition to the Hamiltonian formalism for a theory with higher derivatives. We solve this problem and give (see, Section 6) the explicit form of the canonical transformation which relates the Pais-Uhlenbeck Hamiltonian (with odd frequencies) to the one for the free higher derivatives theory.

2 Conformal mechanics

The prototype of all conformal groups is the one acting in $1+0$ -dimensional spacetime, locally isomorphic to $SL(2, \mathbb{R})$; for the recent developments in conformal mechanics see, e.g., Refs. [26]-[28]. In order to construct the conformal Newton-Hooke dynamics, i.e., the dynamics of the conformal particle in the harmonic trap, we must modify the Hamiltonian by adding the conformal generator. Thus we choose the basis of the $sl(2, \mathbb{R})$ algebra as follows

$$\begin{aligned} [H, D] &= i(H \mp 2K), \\ [D, K] &= iK, \\ [H, K] &= 2iD. \end{aligned} \tag{1}$$

It is worth to note that, although we only change the basis ($H \rightarrow H \pm K$) of $sl(2, \mathbb{R})$ algebra, this alters the dynamics and, consequently, the dynamical realizations of the algebra.

Let us consider the decomposition based on D as the stability subgroup generator. Then the coset space is parametrized as follows

$$w = e^{itH} e^{izK}, \tag{2}$$

and the action of the $SL(2, \mathbb{R})$ group is defined by

$$g e^{itH} e^{izK} = e^{it'H} e^{iz'K} e^{iu'D}, \tag{3}$$

which can be explicitly found by taking the representation spanned by

$$H = i(-\sigma_+ \pm \sigma_-), \quad K = i\sigma_-, \quad D = -\frac{i}{2}\sigma_3. \tag{4}$$

It reads,

$$\begin{aligned} t' &= \arctan \left(\frac{\alpha \tan t + \beta}{\gamma \tan t + \delta} \right), \\ z' &= ((\alpha \sin t + \beta \cos t)^2 + (\gamma \sin t + \delta \cos t)^2) z, \\ &\quad + \frac{1}{2}(\beta^2 + \delta^2 - \alpha^2 - \gamma^2) \sin 2t - (\gamma\delta + \alpha\beta) \cos 2t, \\ u' &= -\ln((\alpha \sin t + \beta \cos t)^2 + (\gamma \sin t + \delta \cos t)^2), \end{aligned} \tag{5}$$

in the oscillatory case (+), and

$$\begin{aligned}
t' &= \operatorname{arctanh} \left(\frac{\alpha \tanh t + \beta}{\gamma \tanh t + \delta} \right), \\
z' &= ((\gamma \sinh t + \delta \cosh t)^2 - (\alpha \sinh t + \beta \cosh t)^2)z, \\
&\quad + \frac{1}{2}(\beta^2 - \delta^2 + \alpha^2 - \gamma^2) \sinh 2t + (\alpha\beta - \gamma\delta) \cosh 2t, \\
u' &= -\ln((\gamma \sinh t + \delta \cosh t)^2 - (\alpha \sinh t + \beta \cosh t)^2),
\end{aligned} \tag{6}$$

in the hyperbolic one (-); here, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})$. Due to Eq. (1) the Cartan forms $w^{-1}dw \equiv i(\omega_H H + \omega_K K + \omega_D D)$ coincide with those in the old basis and read

$$\omega_H = dt \quad \omega_K = dz + z^2 dt, \quad \omega_D = -2z dt. \tag{7}$$

However, the transformation rules change and take the form

$$\omega'_H = e^{u'} \omega_H, \tag{8}$$

$$\omega'_K = e^{-u'} \omega_K + (\pm e^{-u'} \mp e^{u'}) \omega_H, \tag{9}$$

$$\omega'_D = \omega_D - du'.$$

The covariant derivative of z is defined as the ratio of the Cartan forms

$$\nabla z = \frac{\omega_K}{\omega_H} = \dot{z} + z^2; \tag{10}$$

one can easily obtains

$$\nabla z' = e^{-2u'} \nabla z \pm e^{-2u'} \mp 1. \tag{11}$$

In order to construct the invariant dynamics it is sufficient to find the action integral invariant under the dilatation subgroup. This can be easily done by taking the Lagrangian

$$L_0 = \sqrt{\dot{z} + z^2 \pm 1}, \tag{12}$$

or the corresponding Hamiltonian

$$H_0 = \frac{-1}{4p_z} - p_z z^2 \mp p_z, \tag{13}$$

with $\{z, p_z\} = 1$. The Lagrangian (12) (or the Hamiltonian (13)) leads to the following equation of motion

$$\ddot{z} + 6z\dot{z} + 4z^3 \pm 4z = 0. \quad (14)$$

The above equation contains the whole family of conformal models. In fact, with the substitution

$$z = \frac{\dot{\rho}}{\rho}, \quad (15)$$

suggested in Refs. [19] and [29], Eq. (14) yields

$$\frac{d}{dt}(\ddot{\rho}\rho^3 \pm \rho^4) = 0, \quad (16)$$

or

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} \mp \rho, \quad (17)$$

i.e., conformal particle in the harmonic trap.

Let us note that the replacement

$$t \rightarrow it, \quad z \rightarrow -iz, \quad (18)$$

performed in Eq. (14), relates the oscillatory case (+) to the hyperbolic one (−). This, together with the transformation rules described by the first equation (5) and (6) implies the following change of the action of the $SL(2, \mathbb{R})$ group:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow g' = \begin{pmatrix} \alpha & i\beta \\ -i\gamma & \delta \end{pmatrix}. \quad (19)$$

Note that both realizations of $SL(2, \mathbb{R})$ are equivalent but not related by an inner automorphism.

To get rid of the square root in the action integral one can follow the standard procedure by writing

$$L_1 = -\gamma^2\eta - \frac{1}{2\eta}(\dot{z} + z^2 \pm 1), \quad (20)$$

where γ is an arbitrary constant while η is an adjoint field transforming according to $\eta' = e^{-u'}\eta$.

Now, let us perform a simple canonical analysis. The primary constraints read

$$\chi_1 \equiv p_\eta \approx 0, \quad \chi_2 \equiv p_z + \frac{1}{2\eta} \approx 0, \quad (21)$$

while the Hamiltonian is written as

$$H_1 = \gamma^2 \eta + \frac{1}{2\eta}(z^2 \pm 1) + u_\eta p_\eta + u_z(p_z + \frac{1}{2\eta}), \quad (22)$$

u_η, u_z being the appropriate Lagrange multipliers. Imposing

$$\frac{d}{dt}p_\eta \approx 0, \quad \frac{d}{dt}(p_z + \frac{1}{2\eta}) \approx 0, \quad (23)$$

we find no new constraints while

$$u_z = 2\gamma^2 \eta^2 - (z^2 \pm 1), \quad u_\eta = -2z\eta. \quad (24)$$

So the constraints (21) are of the second kind. This allows us to eliminate p_η and p_z at the expense of introducing the Dirac bracket and finally we obtain

$$H_D = \gamma^2 \eta + \frac{z^2 \pm 1}{2\eta}, \quad \{z, \eta\}_D = 2\eta^2. \quad (25)$$

Putting

$$\eta = \frac{1}{\rho^2}, \quad z = \frac{p_\rho}{\rho}, \quad (26)$$

one arrives at the standard form

$$H_D = \frac{1}{2}p_\rho^2 + \frac{\gamma^2}{\rho^2} \pm \frac{1}{2}\rho^2, \quad \{\rho, p_\rho\} = 1. \quad (27)$$

3 Dynamical realizations of the l -conformal Newton-Hooke algebras

The l -conformal Newton-Hooke algebra (in three-dimensional case) is spanned by the generators H, D, K satisfying (1) together with $so(3)$ generators J_k

and $2l + 1$ additional generators $\vec{C}^{(n)}$, $n = 0, 1, \dots, 2l$ obeying

$$\begin{aligned} [H, \vec{C}^{(n)}] &= i(n\vec{C}^{(n-1)} \pm (n-2l)\vec{C}^{(n+1)}), \\ [K, \vec{C}^{(n)}] &= i(n-2l)\vec{C}^{(n+1)}, \\ [D, \vec{C}^{(n)}] &= i(n-l)\vec{C}^{(n)}, \\ [J_i, C_k^{(n)}] &= i\varepsilon_{ikm}C_m^{(n)}. \end{aligned} \tag{28}$$

Consider the nonlinear action defined by selecting the subgroup generated by \vec{J} and D . With such a choice we are not dealing with the symmetric decomposition. However, the generators H, K and $\vec{C}^{(n)}$ span the linear representation under the adjoint action of the stability subgroup. Therefore, our realization linearizes on it. In order to construct the invariant dynamics it is sufficient to respect the invariance under rotations and dilatation.

Let us choose the following parametrization of the coset manifold

$$w = e^{itH} e^{i\vec{x}^{(n)}\vec{C}^{(n)}} e^{izK}, \tag{29}$$

note the difference with respect to the parametrization used in [20]. The Cartan forms

$$w^{-1}dw = i(\omega_H H + \omega_D D + \omega_K K + \vec{\omega}^{(n)}\vec{C}^{(n)}), \tag{30}$$

are given by Eqs. (7) together with

$$\vec{\omega}^{(n)} = \sum_{p=0}^n \binom{2l-p}{2l-n} (-z)^{n-p} (d\vec{x}^{(p)} - (p+1)\vec{x}^{(p+1)}dt \mp (p-1-2l)\vec{x}^{(p-1)}dt). \tag{31}$$

The forms $\vec{\omega}^{(n)}$ are vectors under $SO(3)$ while under dilatation

$$\vec{\omega}'^{(n)} = e^{(l-n)u'} \vec{\omega}^{(n)}. \tag{32}$$

Define the covariant derivatives

$$\nabla \vec{x}^{(n)} \equiv \frac{\vec{\omega}^{(n)}}{\omega_H}, \tag{33}$$

with the dilatation dimension $l - n - 1$. Let $\vec{\lambda}^{(n)}$ be additional (adjoint) variables with dilatation dimension $n - l$. Consider the following first order

Lagrangian

$$L = -\gamma^2 \eta - \frac{1}{2\eta}(\dot{z} + z^2 \pm 1) + \sum_{n=0}^{2l} \vec{\lambda}^{(n)} \nabla \vec{x}^{(n)}. \quad (34)$$

By the very construction it yields the invariant action functional. The equations of motion are of the form

$$\begin{aligned} 2\gamma^2 \eta^2 - (\dot{z} + z^2 \pm 1) &= 0, \\ \dot{\eta} + 2z\eta &= 0, \\ \dot{\vec{x}}^{(n)} - (n+1)\vec{x}^{(n+1)} \mp \vec{x}^{(n-1)}(n-1-2l) &= 0, \quad n = 0, \dots, 2l, \\ \sum_{n=0}^{2l-p} \binom{2l-p}{n} \frac{d}{dt} \left((-z)^n \vec{\lambda}^{(n+p)} \right) + p \sum_{n=0}^{2l-p+1} \binom{2l-p+1}{n} (-z)^n \vec{\lambda}^{(n+p-1)} \\ \pm (p-2l) \sum_{n=0}^{2l-p-1} \binom{2l-p-1}{n} (-z)^n \vec{\lambda}^{(n+p+1)} &= 0. \end{aligned} \quad (35)$$

We see that they decouple. The first two describe the conformal mechanics in the harmonic trap. Then, there is a set of equations for $\vec{x}^{(n)}$ describing higher derivatives system. Let us note that in our approach we do not need to perform the redefinition of time as in Ref. [20]. Finally, once $z(t)$ is determined one can solve the last equation for $\vec{\lambda}^{(n)}$; they do not impose any further constraints on z .

Finally, let us note that extending the transformation rules (18) by

$$\eta \rightarrow -i\eta, \quad \vec{\lambda}_p \rightarrow i^p \vec{\lambda}_p, \quad \vec{x}_p \rightarrow (-i)^p \vec{x}_p, \quad (36)$$

one can transform the Lagrangian (34) from the oscillatory to the hyperbolic case.

Our Lagrangian, being of the first order, provides an example of a constrained system. Following [30], the Hamiltonian dynamics is given by

$$H = \gamma^2 \eta + \frac{z^2 \pm 1}{2\eta} + \sum_{n=0}^{2l} \vec{\lambda}^{(n)} \sum_{p=0}^n \binom{2l-p}{2l-n} (-z)^{n-p} \left((p+1) \vec{x}^{(p+1)} \pm (p-1-2l) \vec{x}^{(p-1)} \right), \quad (37)$$

together with

$$\begin{aligned}
\{x_a^{(n)}, \lambda_b^{(m)}\}_D &= z^{n-m} \binom{2l-m}{2l-n} \delta_{ab}, \\
\{z, \eta\}_D &= 2\eta^2, \\
\{\vec{\lambda}^{(k)}, \eta\}_D &= 2(2l-k)\eta^2 \vec{\lambda}^{(k+1)}.
\end{aligned} \tag{38}$$

Again, it is straightforward, although slightly tedious, to check that Eqs. (37) and (38) yield the correct dynamics.

4 Pais-Uhlenbeck oscillator

It has been shown in Ref. [25] that for l half-integer (i.e. $2l$ is odd) the Pais-Uhlenbeck oscillator of order $2l+1$ [31]

$$L = \frac{(-1)^{l+\frac{1}{2}}}{2} \vec{x} \prod_{k=1}^{l+\frac{1}{2}} \left(\frac{d^2}{dt^2} + \omega_k^2 \right) \vec{x}, \tag{39}$$

with odd frequencies $\omega_k = (2k-1)\omega = (2k-1)$ (in what follows we put $\omega = \frac{1}{R} = 1$), $k = 1, 2, \dots, l + \frac{1}{2}$ enjoys l -conformal Newton-Hooke symmetry (in fact, it is the maximal symmetry group).

In order to compare this finding with our results, let us note that for the l half-integer, we can put the oscillator system defined by the decoupled equations for \vec{x} 's

$$\dot{\vec{x}}^{(n)} - (n+1)\vec{x}^{(n+1)} \mp \vec{x}^{(n-1)}(n-1-2l) = 0, \quad n = 0, \dots, 2l, \tag{40}$$

into the unconstrained Hamiltonian form. To see this we define

$$\begin{aligned}
H &= \sum_{k=0}^{l-\frac{3}{2}} \vec{p}_k \vec{q}_{k+1} + \frac{1}{2} \vec{p}_{l-\frac{1}{2}}^2 \pm \left(- \sum_{k=0}^{l-\frac{3}{2}} (k+1)(2l-k) \vec{q}_k \vec{p}_{k+1} \right. \\
&\quad \left. + \frac{1}{2} \left(l + \frac{1}{2} \right)^2 \vec{q}_{l-\frac{1}{2}}^2 \right),
\end{aligned} \tag{41}$$

which corresponds to our change of the basis $H \rightarrow H \pm K$ in the algebra of the free theory; the standard Poisson brackets read

$$\{q_{ka}, p_{jb}\} = \delta_{kj} \delta_{ab}. \tag{42}$$

The Hamiltonian (41) together with the Poisson brackets (42) yield the following equations of motion

$$\begin{aligned}
\dot{\vec{q}}_k &= \vec{q}_{k+1} \mp (2l+1-k)k\vec{q}_{k-1}, \\
\dot{\vec{p}}_k &= \pm(k+1)(2l-k)\vec{p}_{k+1} - \vec{p}_{k-1}, \\
\dot{\vec{q}}_{l-\frac{1}{2}} &= \vec{p}_{l-\frac{1}{2}} \mp (l+\frac{3}{2})(l-\frac{1}{2})\vec{q}_{l-\frac{3}{2}}, \\
\dot{\vec{p}}_{l-\frac{1}{2}} &= \mp(l+\frac{1}{2})^2 \vec{q}_{l-\frac{1}{2}} - \vec{p}_{l-\frac{3}{2}},
\end{aligned} \tag{43}$$

for $k = 0, \dots, l - \frac{3}{2}$; which after making the substitution

$$\begin{aligned}
\vec{q}_k &= k! \vec{x}^{(k)}, \\
\vec{p}_k &= (-1)^{l-\frac{1}{2}-k} (2l-k)! \vec{x}^{(2l-k)},
\end{aligned} \tag{44}$$

for $k = 0, \dots, l - \frac{1}{2}$, become equivalent to Eqs. (40).

On the other hand, let us observe (see, [32] and [33]) that the Hamiltonian (41) is related (in the (+) case) through a canonical transformation to the one for the Pais-Uhlenbeck Lagrangian (39), i.e.,

$$H = \sum_{k=1}^{l+\frac{1}{2}} \frac{(-1)^{l+\frac{1}{2}-k}}{2} \left(\vec{P}_k^2 + (2k-1)^2 \vec{Q}_k^2 \right). \tag{45}$$

So, in the case of l half-integer there exists an alternative Hamiltonian formalism with no additional variables. On the contrary, for l integer the auxiliary dynamical variables $\vec{\lambda}$'s are necessary.

5 Generalized Niederer's transformation

As it was mentioned before, the l -conformal Newton-Hooke algebra is a counterpart of the l -conformal Galilei one in the presence of a universal cosmological repulsion or attraction. Since these algebras are isomorphic we expect that their dynamical realizations should be related to each other in analogy to the case of $l = \frac{1}{2}$, where their realizations (motion of the free particle and a half-period motion of the harmonic oscillator) are related by famous Niederer's transformation [22]. In this section we will show that the realizations obtained in the preceding sections are also related, by a counterpart of

Niederer's transformation, to the ones obtained in [21] for the l -conformal Galilei algebra.

It is worth to notice that this fact holds for both l integer and half-integer. However, in the second case (as we saw in the preceding section) we have at our disposal an alternative Hamiltonian formalism – the Pais-Uhlenbeck Hamiltonian with odd frequencies. In the next section, we will apply the results obtained here to that important case.

First, let us denote with tilde the dynamical variables entering the realizations of the l -conformal Galilei algebra¹ and define

$$\tilde{\kappa}(\tilde{t}) = \begin{cases} \sqrt{1 + \tilde{t}^2} & (+) \text{ oscillatory case,} \\ \sqrt{1 - \tilde{t}^2} & (-) \text{ hyperbolic case.} \end{cases} \quad (46)$$

Then $\tilde{\kappa}$ satisfies the following useful relations

$$\dot{\tilde{\kappa}}\tilde{\kappa} = \pm\tilde{t}, \quad \dot{\tilde{\kappa}}^2 = \pm 1 \mp \frac{1}{\tilde{\kappa}^2}, \quad \ddot{\tilde{\kappa}} = \pm \frac{1}{\tilde{\kappa}} - \frac{\dot{\tilde{\kappa}}^2}{\tilde{\kappa}}, \quad (47)$$

and, consequently, the equation of motion for the conformal mechanics $\ddot{\tilde{\kappa}} = \pm \frac{1}{\tilde{\kappa}^3}$.

Now, we define a counterpart of Niederer's transformations as follows

$$\begin{aligned} t &= \arctan \tilde{t}, \quad (+) \text{ case}; \quad t = \operatorname{arctanh} \tilde{t}, \quad (-) \text{ case}; \\ z &= \tilde{\kappa}^2 \tilde{z} - \dot{\tilde{\kappa}} \tilde{\kappa}. \end{aligned} \quad (48)$$

First, by the direct calculations, we can check that the action of the $SL(2, R)$ group on (t, z) (Eqs. (5) and (6)) transforms into the one for (\tilde{t}, \tilde{z}) , (cf. Ref. [21]). Next, we verify that the Lagrangian (12) transforms exactly (no total time derivative is needed) into the one obtained in [21], i.e.,

$$\tilde{L}_0 = \sqrt{\dot{\tilde{z}} + \tilde{z}^2}. \quad (49)$$

The same situation occurs on the Hamiltonian level. Indeed, defining

$$p_z = \frac{\tilde{p}_z}{\tilde{\kappa}^2}, \quad (50)$$

¹However, for simplicity, the derivatives with respect to \tilde{t} are also denoted by dots.

we obtain the time dependent canonical transformation, which transforms the Hamiltonian (13) into the conformal one, i.e.,

$$H_0 \frac{dt}{d\tilde{t}} + \frac{\partial F}{\partial t} = \frac{-1}{4\tilde{p}_z} - \tilde{p}_z \tilde{z}^2 = \tilde{H}_0, \quad (51)$$

where

$$\frac{dt}{d\tilde{t}} = \frac{1}{\tilde{\kappa}^2}, \quad (52)$$

while $F_0(z, \tilde{p}_z, \tilde{t}) = \tilde{p}_z(z\tilde{\kappa}^{-2} + \dot{\tilde{\kappa}}\tilde{\kappa}^{-1})$ is the generating function for the transformation (48) and (50).

Moreover, adding the following transformation rule for the dynamical variable η

$$\eta = \tilde{\kappa}^2 \tilde{\eta}, \quad (53)$$

we obtain the generalization of Niederer's transformation for the Lagrangian (20).

Next, we supply the transformations (48) and (53) by the ones for the remaining dynamical variables

$$\begin{aligned} \vec{x}^{(p)} &= \sum_{m=0}^p \binom{2l-m}{2l-p} (-\dot{\tilde{\kappa}})^{p-m} \tilde{\kappa}^{m+p-2l} \tilde{x}^{(m)}, \\ \vec{\lambda}^{(p)} &= \tilde{\kappa}^{2l-2p} \tilde{\lambda}^{(p)}, \end{aligned} \quad (54)$$

where $p = 0, \dots, 2l$. Now, making the substitution defined by Eqs. (48), (53) and (54) in the Lagrangian (34) and using Eqs. (47) together with the following identities

$$\begin{aligned} 0 &= (m-p) \binom{2l-m}{2l-p} + (2l-p+1) \binom{2l-m}{2l-p+1}, \\ 0 &= m \binom{2l-m+1}{2l-p} - (p+1) \binom{2l-m}{2l-p-1} + (2l-m-p) \binom{2l-m}{2l-p}, \end{aligned} \quad (55)$$

we arrive, after straightforward but rather tedious computations, at the Lagrangian invariant under the action of the l -conformal Galilei algebra (see, [21])

$$\tilde{L} = -\gamma^2 \tilde{\eta} - \frac{1}{2\tilde{\eta}} (\dot{\tilde{z}} + \tilde{z}^2) + \sum_{n=0}^{2l} \sum_{p=0}^n \tilde{\lambda}^{(n)} \binom{2l-p}{2l-n} (-\tilde{z})^{n-p} \left(\dot{\tilde{x}}^{(p)} - (p+1) \tilde{x}^{(p+1)} \right). \quad (56)$$

Let us stress that there is no total time derivative entering the transformation rule.

6 Niederer's transformation for Pais-Uhlenbeck model on the Hamiltonian level

Let us recall (see, Ref. [25]) that the Pais-Uhlenbeck oscillator described by the Lagrangian (39) is related to the free higher derivatives theory, defined by the Lagrangian

$$\tilde{L} = \frac{1}{2} \left(\frac{d^{l+\frac{1}{2}} \tilde{x}}{d^{l+\frac{1}{2}} \tilde{t}} \right)^2. \quad (57)$$

The relevant transformation reads

$$t = \arctan \tilde{t}, \quad \vec{x} = \tilde{\kappa}^{-2l} \tilde{\vec{x}}. \quad (58)$$

However, passing to the Hamiltonian counterpart of this transformation we encounter some difficulties; there is no straightforward transition to the Hamiltonian formalism for Lagrangians with higher derivatives (in general, we have to introduce some auxiliary variables and next apply the Dirac's method for constraint systems). We will fill this gap below. Namely, using the results from the preceding sections, we construct a canonical transformation relating the Hamiltonian (41) to the one corresponding to the free theory, i.e., the Ostrogradski Hamiltonian corresponding to the Lagrangian (57):

$$\tilde{H} = \sum_{k=0}^{l-\frac{3}{2}} \tilde{p}_k \tilde{q}_{k+1} + \frac{1}{2} \tilde{p}_{l-\frac{1}{2}}^2. \quad (59)$$

We will work in terms of the variables q 's and p 's and Hamiltonian (41) since in this approach the Pais-Uhlenbeck Hamiltonian (for odd frequencies) is the sum of the Hamiltonian and the conformal generator (at time zero) of the free theory which perfectly corresponds with the relation between the l -conformal Galilei and Newton-Hook algebra. An explicit form of the canonical transformation between q 's and p 's and the decouple harmonic variables as well as Ostrogradski ones will be given in the forthcoming paper [33]; what enables to find this transformation in both remaining approaches.

Let us start with the crucial observation that the relations (44) can be used also in the case of the free theory and that Eqs. (54) define Niederer's-type transformation in our Lagrangian formalism (with no total time derivative entering). Following this idea we obtain the transformation

$$\begin{aligned}\vec{q}_k &= \sum_{m=0}^{l-\frac{1}{2}} b_{km} \tilde{q}_m, \\ \vec{p}_k &= \sum_{m=0}^{l-\frac{1}{2}} (b^{-1})_{mk} \tilde{p}_m + \sum_{m=0}^{l-\frac{1}{2}} c_{mk} \tilde{q}_m,\end{aligned}\tag{60}$$

where

$$\begin{aligned}b_{km} &= \frac{k!}{m!} \binom{2l-m}{2l-k} (-\dot{\kappa})^{k-m} \tilde{\kappa}^{m+k-2l}, \\ c_{mk} &= \frac{(2l-k)!}{m!} (-1)^{l-\frac{1}{2}-k} \binom{2l-m}{k} (-\dot{\kappa})^{2l-k-m} \tilde{\kappa}^{m-k}, \\ (b^{-1})_{mk} &= (-1)^{k+m} \tilde{\kappa}^{4l-2m-2k} b_{mk}\end{aligned}\tag{61}$$

and, by definition, $\binom{k}{m} = 0$ if $k < m$. We will check that Eqs. (60) define, on the Hamiltonian level, an analogue (to the classical case $l = \frac{1}{2}$) of Niederer's transformation relating Pais-Uhlenbeck model with odd frequencies and the free higher derivatives theory, i.e.,

$$H \frac{dt}{d\tilde{t}} + \left(\frac{\partial F}{\partial \tilde{t}} \right) = \tilde{H};\tag{62}$$

where F is the generating function for the transformation (60) and both sides are expressed in terms of \tilde{q} 's and \tilde{p} 's².

First, by the standard calculations we check that Eqs. (60) define a canonical transformation. Further, we find the generating function

$$F(\vec{q}_0, \dots, \vec{q}_{l-\frac{1}{2}}, \vec{p}_0, \dots, \vec{p}_{l-\frac{1}{2}}, \tilde{t}) = \sum_{k=0}^{l-\frac{1}{2}} \tilde{p}_k \tilde{q}_k(\vec{q}_0, \dots, \vec{q}_{l-\frac{1}{2}}, \tilde{t}) + \frac{1}{2} \sum_{k,m=0}^{l-\frac{1}{2}} a_{km} \vec{q}_k \vec{q}_m,\tag{63}$$

²Eq. (62) is the well know transformation rule for the Hamiltonian, under a canonical transformation, in the case when time variable.

where

$$a_{km} = \frac{(-1)^{k+m}(2l-k)!(2l-m)!}{k!m!(l-\frac{1}{2}-k)!(l-\frac{1}{2}-m)!} \frac{(-\tilde{\kappa}\dot{\tilde{\kappa}})^{2l-k-m}}{(2l-k-m)} = a_{mk}, \quad (64)$$

and, by virtue of (60)

$$\tilde{q}_m(\vec{q}_0, \dots, \vec{q}_{l-\frac{1}{2}}, \tilde{t}) = \sum_{k=0}^{l-\frac{1}{2}} (b^{-1}(\tilde{t}))_{mk} \vec{q}_k. \quad (65)$$

To this end the identity

$$\sum_{k=0}^a (-1)^k \binom{a+b}{k} = (-1)^a \binom{a+b-1}{a}, \quad 0 \leq a, 1 \leq b, \quad (66)$$

appears to be very useful. Next, we prove Eq. (62): due to the fact that the Pais-Uhlenbeck model is traditionally considered in the oscillatory regime (cf. Eqs. (39) and (45)), we will focus on the (+) case; the (−) case can be treated in the same way or by using the observation that the transformation

$$\vec{q}_k \rightarrow (-i)^{l+\frac{1}{2}+k} \vec{q}_k, \quad \vec{p}_k \rightarrow (-1)^{l+\frac{1}{2}} (-i)^{l-\frac{1}{2}-k} \vec{p}_k, \quad (67)$$

relates (+) and (−) cases.

Using Eq. (47) as well as the known properties of the binomial coefficients we obtain, after straightforward but rather tedious computations, the derivative of F with respect to \tilde{t} – expressed in terms of \tilde{q} 's and \tilde{p} 's:

$$\begin{aligned} \frac{\partial F}{\partial \tilde{t}}(\tilde{q}_0, \dots, \tilde{q}_{l-\frac{1}{2}}, \tilde{p}_0, \dots, \tilde{p}_{l-\frac{1}{2}}, \tilde{t}) &= \frac{1}{\tilde{\kappa}^2} \sum_{m=0}^{l-\frac{3}{2}} (2l-m)(m+1) \tilde{p}_{m+1} \tilde{q}_m \\ &+ \frac{2\dot{\tilde{\kappa}}}{\tilde{\kappa}} \sum_{m=0}^{l-\frac{1}{2}} (l-m) \tilde{p}_m \tilde{q}_m - \frac{1}{2\tilde{\kappa}^2} (l+\frac{1}{2})^2 \tilde{q}_{l-\frac{1}{2}}^2. \end{aligned} \quad (68)$$

So, to prove Eq. (62) it remains to express H in terms of \tilde{q} 's and \tilde{p} 's. The explicit calculations are troublesome so we will sketch only the main steps.

First we find the coefficients in front of the terms $\tilde{q}_m \tilde{q}_{\bar{m}}$. Using Eq. (47) and the identities (55) we derive the following relations

$$(k+1)(2l-k)c_{m,k+1} - c_{m,k-1} = \tilde{\kappa}^2 c_{m-1,k} - 2\tilde{\kappa}\dot{\tilde{\kappa}}(l-m)c_{mk} - (2l-m)(m+1)c_{m+1,k}, \quad (69)$$

for $k, m = 0, \dots, l - \frac{1}{2}$. Next, applying the identity (66), we compute the expressions of the type $\sum_{k=m}^{l-\frac{1}{2}} c_{\bar{m}k} b_{km}$. Due to the symmetry $m \leftrightarrow \bar{m}$ the final result is of the form $\frac{1}{2}(l + \frac{1}{2})^2 \tilde{q}_{l-\frac{1}{2}}^2$, and by Eq. (52) it cancels against the last term of Eq. (68).

To compute the coefficients in front of the terms bilinear in \tilde{q} 's and \tilde{p} 's, we first derive, by virtue of Eqs. (47) and (55) the following identity

$$b_{k+1,m} - k(2l - k + 1)b_{k-1,m} = \tilde{\kappa}^2 b_{k,m-1} - 2\tilde{\kappa}\dot{\tilde{\kappa}}(l - m)b_{km} - (2l - m)(m + 1)b_{k,m+1}, \quad (70)$$

for $m = 0, \dots, l - \frac{1}{2}$ and $k = 0, \dots, l + \frac{1}{2}$. Using (70) and (52) we conclude that the final result contains three terms: two of them cancel against the first two terms of Eq. (68) and there only remains the sum $\sum_{m=0}^{l-\frac{1}{2}} \tilde{p}_m \tilde{q}_{m+1}$. Finally, it is quite easy to check that the only nonvanishing term bilinear in \tilde{p} 's is $\frac{1}{2}\tilde{p}_{l-\frac{1}{2}}^2$. In summary, we obtain the Hamiltonian \tilde{H} (cf. Eq. (59)) and, consequently, the relation (62).

7 Conclusions

We have used the method of the nonlinear realizations to construct dynamical systems invariant under the action of the l -conformal Newton-Hooke algebra for both integer and half-integer values of l . We put emphasis on the Lagrangian and Hamiltonian formulation. Therefore, instead of imposing invariant constraints on the Cartan forms we enlarged the stability subgroup (in order to abandon one constraint) and added new variables which allow us to construct a simple invariant Lagrangian in such a way that these new degrees of freedom do not enter the dynamics of the original ones. The resulting dynamical equations of motion are described by Eqs. (35). The characteristic property of Eqs. (35) is that they decouple. We have achieved this by the appropriate choice of the subgroup, on which the action of the l -conformal group linearizes (rotations and dilatation) and the specific parametrization of the coset manifold (cf. Eq. (29)).

We have shown that this description is universal in the sense that it works whether l is half-integer or integer. The difference between the case of l integer or half-integer is that the latter admits, besides the Hamiltonian formalism presented here, an alternative one where no additional variables are necessary, namely, the Hamiltonian formalism of the Pais-Uhlenbeck oscillator with odd frequencies. Note that, when $\vec{\lambda}^{(n)}$ variables are present, the

group action is no longer transitive and the phase space is not a coadjoint orbit and cannot be directly obtained by the orbit method.

Next, we constructed an analogy of Niederer's transformation relating the dynamics described in Section 2 and 3 to the one constructed in Ref. [21] for the l -conformal Galilei algebra. Moreover, we use this transformation as well as the relations between our Lagrangian formalism and the Pais-Uhlenbeck theory to find the counterpart of Niederer's transformation for the Pais-Uhlenbeck oscillator on the Hamiltonian level. This is accomplished by the canonical transformation (60). We believe that this transformation can be useful to extend Niederer's transformation to the quantum version of the Pais-Uhlenbeck model as well as the study of its quantum symmetries. It is also tempting (especially in the context of the recent results [34]) to extend the present considerations to the supersymmetric case: in particular, to find supersymmetric extensions of Niederer's transformations.

Acknowledgments. Special thanks are to Piotr Kosiński for valuable comments and suggestions. The discussions with Joanna Gonera and Paweł Maślanka are gratefully acknowledged. The work is supported by the grant of National Research Center number DEC-2013/09/B/ST2/02205.

References

- [1] H. Bacry, J.M. LévyLeblond, J. Math. Phys. **9** (1968) 1605.
- [2] G.W. Gibbons, C.E. Patricot, Class. Quant. Grav. **20** (2003) 5225.
- [3] P. Havas, J. Plebański, J. Math. Phys. **19** (1978) 482.
- [4] M. Henkel, J. Stat. Phys. **75** (1994) 1023.
- [5] J. Negro, M.A. del Olmo, A. Rodriguez-Marco, J. Math. Phys. **38** (1997) 3810.
- [6] C. Duval, P.A. Horvathy, J. Phys. A **42** (2009) 465206.
- [7] J. Lukierski, P.C. Stichel, W.J. Zakrzewski, Phys. Lett. A **357** (2006) 1.
- [8] F.L. Liu, Y. Tian, Phys. Lett. A **372** (2008) 6041.
- [9] C. Duval, M. Hassaine, P.A. Horvathy, Ann. Phys. **324** (2009) 1158.

- [10] A. Galajinsky, Nucl. Phys. B **832** (2010) 586.
- [11] J. Gomis, K. Kamimura, Phys. Rev. D **85** (2012) 045023.
- [12] K. Andrzejewski, J. Gonera, P. Maślanka, Phys. Rev. D **86** (2012) 065009.
- [13] K. Andrzejewski, J. Gonera, Phys. Lett. B **721** (2013) 319.
- [14] N. Aizawa, Y. Kimura, J. Segar, J. Phys. A **46** (2013) 405204.
- [15] S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. **177** (1969) 2239.
- [16] C.G. Callan, S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. **177** (1969) 2247.
- [17] E.A. Ivanov, V.I. Ogievetsky, Teor. Mat. Fiz. **25** (1975) 164.
- [18] S. Fedoruk, E. Ivanov, J. Lukierski, Phys. Rev. D **83** (2011) 085013.
- [19] A.V. Galajinsky, I. Masterov, Nucl. Phys. B **866** (2013) 212.
- [20] A. Galajinsky, I. Masterov, Phys. Lett. B **723** (2013) 190.
- [21] K. Andrzejewski, J. Gonera, P. Kosiński, P. Maślanka, Nucl. Phys. B **876** (2013) 309.
- [22] U. Niederer, Helv. Phys. Acta **46** (1973) 191.
- [23] C. Duval, P. Horvathy, J. Phys. A **44** (2011) 335203.
- [24] A. Galajinsky, I. Masterov, Phys. Lett. B **702** (2011) 265.
- [25] K. Andrzejewski, A. Galajinsky, J. Gonera, I. Masterov, Nucl. Phys. B **885** (2014) 150.
- [26] S. Fedoruk, E. Ivanov, O. Lechtenfeld, Journ. Phys. A **45** (2012) 173001.
- [27] A. Gonera, Ann. Phys. **335** (2013) 61.
- [28] N. L. Holanda, F. Toppan, J. Math. Phys. **55** (2014) 061703.
- [29] E.A. Ivanov, S.O. Krivonos, V.M. Leviant, J. Phys. A **22** (1989) 345.

- [30] L. Fadeev, R. Jackiw, Phys. Rev. Lett. **60** (1988) 1692.
- [31] A. Pais, G.E. Uhlenbeck, Phys. Rev. **79** (1950) 145.
- [32] K. Andrzejewski, J. Gonera, Phys. Rev. D **88** (2013) 065011.
- [33] K. Andrzejewski, Hamiltonian formalisms and symmetries of the Pais-Uhlenbeck oscillator, in preparation.
- [34] I. Masterov, preprint, Dynamical realizations of $\mathcal{N} = 1$ l -conformal Galilei superalgebra, arXiv:1407.1438 (2014).